The Hele-Shaw problem as a "Mesa" limit of Stefan problems: Existence, uniqueness, and regularity of the free boundary

Ivan A. Blank
Marianne K. Korten
Charles N. Moore

February 1, 2008

Abstract

We study a Hele-Shaw problem with a mushy region obtained as a mesa type limit of one phase Stefan problems in exterior domains. We study the convergence, determine some of the qualitative properties and regularity of the unique limiting solution, and prove regularity of the free boundary of this limit under very general conditions on the initial data. Indeed, our results handle changes in topology and multiple injection slots.

Key Words: Mesa problem; Hele-Shaw problem; Stefan problem; free boundary; mushy region; singular limit;

2000 Mathematics Subject Classification: 76D27, 35K65, 49J40

1 Introduction

Given a bounded domain $D \in \mathbb{R}^n$ with smooth boundary ∂D, a finite set of closed curves $\{s_0^j(x) = 0\}$ such that D is contained in the union of their interiors, and a continuous function $p(x,t)$ defined on $\partial D \times (0, \infty)$, the classical Hele-Shaw problem with Dirichlet data is frequently formulated as follows: Find a function $V(x,t)$ and a family of domains $S(t)$ (which each contain D)

with $\partial S(t) = \{t = s(x)\}$ and such that

$$\begin{array}{rll} \Delta_x V & = 0 & (x,\ t) \in S(t) \setminus D \\ V(x,\ t) & = 0 & (x,\ t) \in \partial S(t) \\ V(x,\ t) & = p(x) & (x,\ t) \in \partial D \times (0,\ \infty) \\ \nabla_x V \cdot \nabla_x s & = \frac{\partial s}{\partial t} & (x,\ t) \in \partial S(t) \\ \partial S(0) & = \cup_j \{s_0^j(x) = 0\}\ . & \end{array} \tag{1.1}$$

This problem models the advance of the slick formed by injecting oil between two nearby plates, and has further been used in injection molding (used in turn in the packaging industry, and more generally for the production of plastic components, for example interior pieces of cars and aircraft), in electrochemical machining (see [MR]), and even to predict tumor growth (see [BF]). Sometimes the normal derivative of V at the "slot," ∂D, is prescribed, or curvature dependent terms are included in the free boundary condition. Among the most pressing open questions concerning the Hele-Shaw problem are finding a weak formulation, studying the regularity of $V(x,t)$ in t, and determining the regularity of the free boundary $t = s(x)$. Another question which has long attracted interest is whether a Hele-Shaw problem could have a "mushy" region.

In turn, the "Mesa" problem describes the limit pattern $\lim_{m\to\infty} u_m$ of solutions u_m of, say, the porous medium equation, when the initial data are held fixed. This problem first appeared in connection with the modeling of problems related to transistors (see [EHKO]). Caffarelli and Friedman studied the initial value problem in $\mathbb{R}^n \times [0,T]$ in [CF]. They proved that the limit exists, that it is independent of the chosen subsequence, that it is independent of time, that it is equal to the characteristic function of one set plus the initial data times the characteristic function of the complement of that set, and finally that that set can be characterized as the noncoincidence set of a variational inequality. Further developments showed that the same conclusions hold for the limit when u^m is replaced by a fairly general monotone constitutive function $\phi(u)$ with $\phi(0) = 0$ (see [FH] for example), moreover, this behavior is a property of fairly general semigroups (see [I], [BEG]). A Mesa problem for an equation which gives rise to a mushy region was studied in [BKM]. For the Mesa problem we study in this paper we show all of the properties shown by [CF] mentioned above, except that our limits will not be independent of time. (This evolution in time is natural since we work in an outer domain where the inner boundary data serves as a source.)

In this paper the authors exploit a Mesa limit setting in an outer domain D^c to obtain naturally a weak formulation of the Hele-Shaw problem (with Dirichlet condition as in Equation (1.1) on the slot as above). The use of one-phase Stefan problems with with "mushy" regions and with increasing diffusivities naturally produces a mushy region when we permit initial data

u_I for the approximating problems to take values in the interval [0, 1]. These u_I can be thought of as generalized characteristic functions. Another aspect of this approach which is extremely attractive is the fact that changes in the topology of the "wet" region do not interfere with the construction. In short, whereas other authors have taken a priori assumptions which ensure that their free boundary stays smooth, we have been able to show existence of weak solutions for all time, regardless of the possible changes in topology. (Note that in [CF] and [FH] they assume that their data is starlike with respect to the origin, and in [DL] log concavity of initial data is assumed to guarantee existence and smoothness of the solutions.) Indeed there are some very natural problems arising in the applications where the topology should change. Consider for example the problem of what happens with Hele-Shaw flow around an obstacle. In this case, there is automatically a change in topology when the flow meets itself on the other side. An interesting question is whether or not an air bubble will be left behind in the wake of the obstacle. Another obvious problem from applications where there will be changes in topology is if there are multiple injection slots. In fact, this paper already deals with the second situation, since we never assume that D is connected.

The classical version of the m-approximating problem which we use is given as follows:

1.1 Definition (m-approximating problem). We let $u^{(m)}$ denote the solution of the following partial differential equation,

$$u_t^{(m)} = m\Delta(u^{(m)}(x,t) - 1)_+, \qquad (x,t) \in D^c \times (0,\ +\infty) \cap \{u^{(m)} > 1\}, \quad (1.2)$$

with boundary data given by

$$\left.\begin{aligned} u^{(m)}(x,0) &= u_I(x), && x \in D^c, \\ m(u^{(m)}(x,t) - 1)_+ &= p(x), && (x,t) \in \partial D \times (0,\ +\infty), \end{aligned}\right\} \quad (1.3)$$

and with the free boundary condition

$$(-\nabla m(u^{(m)} - 1)_+, (1 - u_I)\ ) \cdot \nu = 0\ , \quad (x,t) \in \partial\{u^{(m)} > 1\}\ . \quad (1.4)$$

Here ν is the outer (n + 1) dimensional normal to the set $\{u^{(m)} > 1\}$, and this free boundary condition will be satisfied when the free boundary is smooth. The weak formulation we give at the beginning of the next section will not require any regularity assumptions on the free boundary or on the initial data. For a fixed $m > 0$, we call the free boundary problem determined by the equations above the *m-approximating problem*. We will assume

$$\begin{aligned} &0 \le u_I \le 1 \text{ is compactly supported,} \\ &0 < p(x) \in C^{2,\alpha}(\partial D)\ , \\ &D \text{ is a bounded set with } \partial D \in C^{2,\alpha}. \end{aligned} \quad (1.5)$$

1.2 Remark (Simplifying Assumptions). There are two assumptions which we make which serve to simplify the exposition but which are absolutely *not* necessary for the derivation of our results:

1. The assumption that u_I has compact support is only technical and serves to simplify the construction of certain comparison functions in the fifth section. We leave to the reader the verification that it suffices to assume only that $\{u_I = 1\}$ is compact for all of the results before the sixth section.

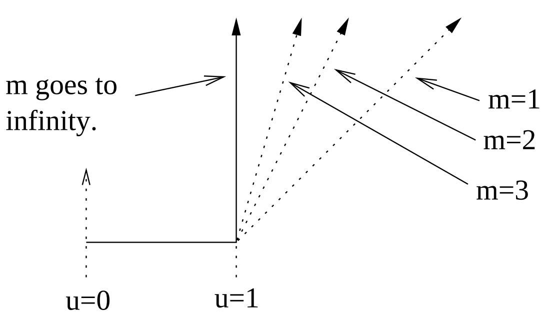
2. The assumption that the Dirichlet data, $p(x)$, on the boundary of the slot is a function of x alone can be replaced with the assumption that the Dirichlet data $p(x,t)$ is a nondecreasing function of t for each fixed x. On the other hand, if $p(x,t)$ is allowed to decrease in time, then Lemma (4.1) does not hold in general, and this appears to be essential for the methods presented in this paper.

For convenience, we will define

$$M := ||p||_{L^\infty(\partial D)} \ . \tag{1.6}$$

Now in terms of the positivity assumption on $p(x)$, we note that $p(x) \equiv 0$ leads to a trivial case: There will be no evolution at all. To see this fact extend each $u^{(m)}$ by 1 across all of D. In short, the positivity of $p(x)$ is driving the evolution.

In this formulation $u^{(m)}$ is energy (or enthalpy), and $m(u^{(m)}(x,t)-1)_+$ is temperature. Equation (1.2) comes from conservation of energy. Basically, as m increases, the diffusion happens faster. Competing with this increase in diffusion is the fact that the boundary data for $u^{(m)}$ on ∂D is decreasing down to 1. As $m \to \infty$ we have convergence of our operators to the following picture which is typical for Mesa problems:



We show that as $m \to \infty$, the $u^{(m)}$ converge pointwise to a limit $0 \leq u^{(\infty)} \leq 1$, and $m(u^{(m)}-1)_+$ tend pointwise to a bounded function $V(x,\ t)$. Furthermore, the function V is identically zero in $\{u^{(\infty)} < 1\}$ and positive and

harmonic for fixed t in the component of the set $\{u^{(\infty)} = 1\}$ which contains D. (On sets where $\{u^{(\infty)} = 1\}$ which are isolated from D we will have $V \equiv 0$, see Remark (3.1).) Finally, the pair $(u^{(\infty)},\ V)$ solves our weak Hele-Shaw problem with a mushy region which we give formally in Definition (2.6). In this way we obtain the following results.

1. We get a natural generalization of the weak formulation of the Hele-Shaw problem of DiBenedetto and Friedman (see [DF]). Our formulation allows for a "mushy" region. Moreover, after invoking a result of Bouillet, we will be able to say that solutions of our formulation are unique, and hence the Mesa limiting procedure we use gives an effective method of constructing the solution.

2. We determine the regularity of the spatial slices of the free boundary by invoking the results of Blank, Caffarelli, and Kinderlehrer and Nirenberg for the regularity of the free boundary in the obstacle problem (see [Bl], [C], [KN]). Indeed, if we define

$$U(x,t) := \int_0^t u_t^{(\infty)}(x,\ s)\ ds = (1 - u_I(x))\chi_{_{A(t)}}(x) \tag{1.7}$$

   where $A(t)$ gives the "puddle" at time t, and

$$W(x,t) := \int_0^t V(x,\ s)\ ds\ , \tag{1.8}$$

   and we formally apply the Baiocchi transformation to the equation

$$u_t^{(\infty)}(x,\ t) = \Delta_x V(x,\ t)\ , \tag{1.9}$$

   then for every fixed time t_0, W satisfies the obstacle problem:

$$0 \le W(x,t_0)\ , \qquad \Delta_x W(x,t_0) = U(x,t_0)\ . \tag{1.10}$$

In a subsequent paper the authors will address the rectifiability and further space-time regularity of the free boundary.

The paper is arranged as follows: In section 2 we introduce our notion of weak solutions to the approximating problem, show some qualitative properties of these solutions, derive the existence of the limits (for now in weak-$*$ L^∞), and give some trivial bounds on these limits. Most of the results in this section draw from the maximum principle and from the papers [AK], [K1], [K2], and [K3]. Section 3 gives a simple counter-example which motivates the definition of the free boundary and of the diffusive region. In section 4 we derive some monotonicity properties of our sequences and limits, and as consequences we improve our weak-$*$ L^∞ convergence to pointwise convergence and give an

explicit representation of $u^{(\infty)}$. In section 5 we show that the limiting problem has a free boundary for all finite time (as opposed to if it "escaped to infinity" in zero time or by a fixed time), and we use explicit subsolutions to show that if an open ball within the diffusive region has a boundary point on the free boundary, then the free boundary will have nonzero velocity at that point. In section 6 we apply the Baiocchi transformation to Equation (1.9) and thereby derive a family of obstacle problems that this procedure yields. At that point, under suitable assumptions on the initial data, we can invoke the regularity theory for the obstacle problem (see [Bl] and [C]) to immediately derive regularity in space for the free boundary of our Hele-Shaw problem for almost every time. In section 7, we extend the results of section 6 to include every time. In the process of extending to every time, we establish the continuity of the measure of the diffusive region with respect to time when the initial data is strictly less than one. Section 8 is an appendix which gives some of the barrier functions and some of their properties that we need in some of the earlier sections.

I. Blank wishes to thank Kansas State University for their hospitality while the bulk of this paper was written. M. Korten is indebted to Juan Luis Vázquez for introducing her to the Hele-Shaw problem. M. Korten was partially supported by NSF EPSCoR Grant #530517 under agreement NSF32169/KAN32170, and I. Blank's visits to Kansas State University were also supported by this grant. All three authors wish to thank Barbara Korten for her patience and support while this paper was written.

2 The weak formulations

For our m-approximating problem, we need an appropriate weak formulation which we give here:

2.1 Definition (Weak solutions of the m-approximating problem). The nonnegative function $u^{(m)}(x,\ t) \in L^1_{loc}$ is a weak solution of the m-approximating problem if for any $\varphi \in C^\infty(\mathbb{R}^n \times [0,\infty))$, such that $\varphi \equiv 0$ on $\partial D \times [0,\infty)$, and $\varphi(x,t) \to 0$ as either $t \to \infty$ or $|x| \to \infty$ we have

$$\int_{D^c}\int_0^\infty \varphi_t(x,t)u^{(m)}(x,t)\ dt\ dx + \int_{D^c}\int_0^\infty \Delta_x\varphi(x,t)m\left[u^{(m)}(x,t)-1\right]_+\ dt\ dx$$
$$= \int_{\partial D}\int_0^\infty \frac{\partial\varphi}{\partial\nu}(x,t)p(x)\ dt\ d\mathcal{H}^{n-1}x - \int_{D^c}\varphi(x,0)u_I(x)\ dx\ . \tag{2.1}$$

We observe that the traces of $u^{(m)}$ and $m(u^{(m)}-1)_+$ on the boundaries of our domain will be well-defined by the work of Korten even if we only assume that we are dealing with local solutions and that the initial trace is between

zero and one. (See [K1] which adapts the work of Dahlberg and Kenig, [DK], and see Lemma 3.2 of [K2].) Furthermore, we observe that the free boundary condition for the classical formulation (i.e. Equation (1.4)) will be satisfied, whenever the functions and sets are sufficiently smooth.

For strictly local situations, it may be helpful to note that if $\varphi \in C_0^\infty(\Omega)$ where $\Omega \subset\subset D^c \times (0, +\infty)$, then we have

$$\begin{aligned}
&\int_{D^c} \int_0^\infty \varphi_t(x,t) u^{(m)}(x,t)\, dt\, dx \\
&= - \int_{D^c} \int_0^\infty m \Delta_x \varphi(x,t) \left[u^{(m)}(x,t) - 1\right]_+ \, dt\, dx \\
&= + \int_{D^c} \int_0^\infty m \nabla_x \varphi(x,t) \cdot \nabla_x \left[u^{(m)}(x,t) - 1\right]_+ \, dt\, dx \, .
\end{aligned}$$

The last integration by parts requires that we invoke the known regularity theory and energy estimates for solutions of Equation (1.2). See Lemma 1.2 of [AK].

Since we want to discuss regularity of functions in Sobolev spaces, we need to fix ideas about which representative we will use. Although the "Lebesgue point" representative where points are filled in by taking limits of averages over balls with radii going to zero is the most common procedure, we will use a slightly different approach which exploits the fact that the functions $u^{(m)}(x,t)$ are nondecreasing in t for almost every x by Lemma 4.2 of [K1] and by the fact that the boundary data $p(x,t)$ is nondecreasing in time for each fixed x.

2.2 Definition (Representative). For $t > 0$, we set

$$\tilde{u}^{(m)}(x,\ t) := \lim_{s \uparrow t} u^{(m)}(x,\ s) \quad \text{for } a.e.\ x.$$

For economy of notation we will not bother to relabel any of our $u^{(m)}$'s, but always understand that we are using $\tilde{u}^{(m)}$ as our representative for $u^{(m)}$ in all pointwise matters.

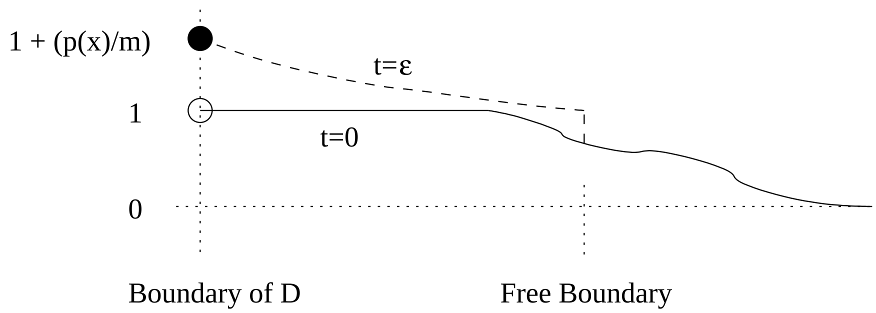
We cannot expect $u^{(m)}$ to be continuous across the free boundary, and in fact, the following theorem summarizes the results of Lemma 2.4 and Lemma 2.5 of [K3]:

2.3 Theorem ($u^{(m)}$ must jump). *The set*

$$E := \{x \in \mathbb{R}^n \ :\ \exists\ t\ s.t.\ u_I(x) < u^{(m)}(x,\ t) < 1\}$$

has Lebesgue n-dimensional measure equal to zero.

On the other hand, $m(u^{(m)}-1)_+$ will be continuous across the free boundary. (See [K1] which verifies the required assumptions of [DB] for continuity.) For a fixed m we now have the following picture of $u^{(m)}$ viewed from the side:



By using the boundedness of the data along with the maximum principle that the equation above enjoys, we can conclude that the solutions are bounded and therefore by elementary functional analysis we can conclude that the following limit

$$u^{(\infty)} := \lim_{m\to\infty} u^{(m)}(x,t), \tag{2.2}$$

exists weak-$*$ L^∞ along a subsequence of m in the entire domain. Again by using the maximum principle it is immediate that

$$0 \le u^{(\infty)} \le 1 \; . \tag{2.3}$$

In fact, by using the maximum principle again, we can assert that

$$0 \le u^{(m)} \le 1 + \frac{M}{m} \qquad \text{or} \qquad 0 \le m(u^{(m)}-1)_+ \le M \; , \tag{2.4}$$

and we stress that M is independent of m and t. Because of this fact, we can take a further subsequence to ensure that $m(u^{(m)}-1)_+$ has a limit V in the weak-$*$ L^∞ topology of the entire domain.

2.4 Remark. By combining Equation (2.4) with Theorem (2.3) we can conclude that the essential range of $u^{(m)}(x,\;\cdot)$ is a subset of $\{u_I(x)\}\cup[1,\;1+M/m]$.

Next, for $\varphi \in H_0^1(\Omega)$ we have:

$$\int_0^{+\infty}\int_{D^c} \nabla_x \left[m(u^{(m)}-1)_+\right]\cdot\nabla_x\varphi \; dx\; dt = \int_0^{+\infty}\int_{D^c} u^{(m)}\varphi_t \; dx\; dt$$
$$\to \int_0^{+\infty}\int_{D^c} u^{(\infty)}\varphi_t \; dx\; dt \; .$$

Since $u^{(\infty)} \in L^\infty(\Omega) \subset L^2(\Omega)$, we have

$$u_t^{(\infty)} \in H^{-1}(\Omega) \tag{2.5}$$

We summarize with the following lemma whose proof is now obtained trivially by using the weak-$*$ L^∞ compactness we have and taking the limits as $m \to \infty$ in the weak formulation of the m-approximating problem.

2.5 Lemma (Limits solve the limiting problem). *Under our assumptions as above, we have*

$$0 \le V \le M \;, \tag{2.6}$$

and the pair $(u^{(\infty)},\ V)$ *satisfies*

$$\begin{aligned} &\int_{D^c}\int_0^\infty \varphi_t(x,t)u^{(\infty)}(x,t)\;dt\;dx + \int_{D^c}\int_0^\infty \Delta_x\varphi(x,t)V(x,t)\;dt\;dx \\ = \;& \int_{\partial D}\int_0^\infty \frac{\partial\varphi}{\partial\nu}(x,t)p(x)\;dt\;d\mathcal{H}^{n-1}x - \int_{D^c}\varphi(x,0)u_I(x)\;dx\;. \end{aligned} \tag{2.7}$$

for any $\varphi \in C^\infty\left(\mathbb{R}^n \times [0,\infty)\right)$, *such that* $\varphi \equiv 0$ *on* $\partial D \times [0,\infty)$, *and* $\varphi(x,t) \to 0$ *as either* $t \to \infty$ *or* $|x| \to \infty$.

2.6 Definition (Hele-Shaw with mushy region). Any pair $(u^{(\infty)},\ V)$ which satisfies Equation (2.7) for φ as in the lemma above will be called a weak solution of the Hele-Shaw problem with boundary data $p(x)$ and initial data $u_I(x)$.

2.7 Remark (Inclusion of earlier models). The weak formulation we have above generalizes the Hele-Shaw formulation of DiBenedetto and Friedman to include the situation where there is a mushy region (see [DF]). In addition, our a priori regularity assumptions on the solutions are weaker. (We only assume L^1_{loc}.)

2.8 Theorem (Uniqueness among all solutions). *The solution of the limiting problem of Lemma (2.5) is unique, so all solutions of the Hele-Shaw problem as given in Definition (2.6) are recoverable via the Mesa limit process we have introduced.*

Proof. This theorem is an immediate application of the uniqueness theorem in section 3 of [Bo].
Q.E.D.

3 Counter-example to regularity in time

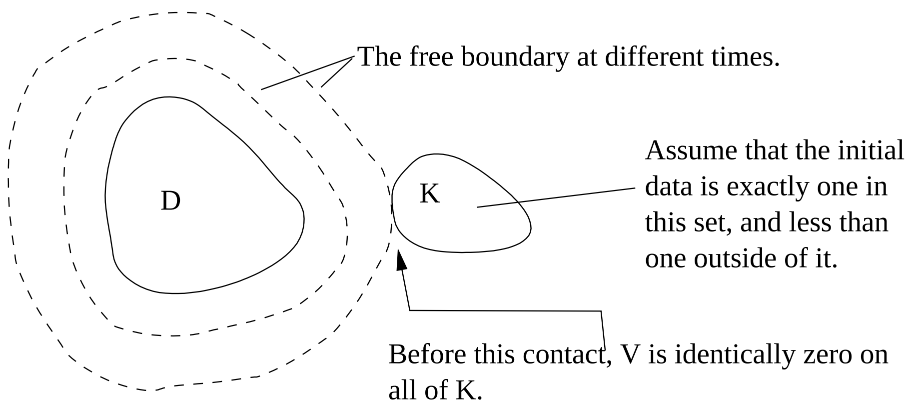
Based on the assumptions we have so far, the function $V(x,t)$ will *not* be continuous in time in general. To show this, we will assume for the sake of this example that u_I is a continuous function. We will also assume in this

section that the set D is connected. The trouble appears in certain cases where the set

$$W := D \cup \{x \in \mathbb{R}^n : u_I(x) = 1\} \tag{3.1}$$

is disconnected. If K is a component of the set $\{u_I = 1\}$ and K is a positive distance away from D, then the $u^{(m)}$ and therefore the $u^{(\infty)}$ should not evolve on this set of x until the free boundary comes into contact with it. Essentially, the "patch" K will not "see" the input of the slot until the component of the set $\{u^{(m)} \geq 1\}$ which surrounds D connects to it.

3.1 Remark (V is not always continuous). To produce a situation where V must be discontinuous in time, simply consider the radially symmetric situation where $D = B_1$ and $u_I(r)$ is taken such that it is identically one on the set $3 \leq r \leq 5$, but smaller outside of it. When what we want to call the free boundary reaches $r = 3$, then it will instantaneously jump to $r = 5$, and this leads to an immediate jump in the height of V.



With these pictures in mind we make the following definitions:

3.2 Definition (Defining the free boundary). We set

$$\begin{aligned} t_m(x) &:= \inf\{t \ : \ m(u^{(m)}(x,t) - 1)_+ > 0\} \\ t_\infty(x) &:= \inf\{t \ : \ V(x,t) > 0\} \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \tau_m(x) &:= \inf\{t \ : \ u^{(m)}(x,t) \geq 1\} \\ \tau_\infty(x) &:= \inf\{t \ : \ u^{(\infty)}(x,t) = 1\} \ . \end{aligned} \tag{3.3}$$

We can also now define the diffusive region at time t for the m-approximating problem and the limiting Hele-Shaw problem to be respectively,

$$\begin{aligned} A^{(m)}(t) &:= \{x \in D^c : \ m(u^{(m)}(x, \ t) - 1)_+ > 0\} \\ A(t) &:= \{x \in D^c \ : \ V(x, \ t) > 0\} \ . \end{aligned} \tag{3.4}$$

Now we simply define the free boundary at time t to be $FB^{(m)}(t) := \partial A^{(m)}(t) \setminus \partial D$, or $FB(t) := \partial A(t) \setminus \partial D$, according to which problem we are considering. (Here "∂K" denotes the topological boundary of K.)

Observe that because of the radially symmetric example given above where V is discontinuous in time, it is also clear that the free boundary cannot be expected to vary continuously in time.

3.3 Remark ("Diffusive" versus "Wet"). The reader may observe that "Wet" may be a more appropriate name for what we are calling the diffusive region in the Hele-Shaw problem. On the other hand, "wet" might more properly describe any region where $u^{(\infty)}(x,t) > 0$, so we will stick to what is already appropriate in the approximating problems.

By considering the figure above, it is clear that in general $t_m(x)$ does not have to equal $\tau_m(x)$, and similarly with $t_\infty(x)$ and $\tau_\infty(x)$. In the figure, they would differ on the set K. For $x \in K$ we have $\tau_m(x) = \tau_\infty(x) = 0$, while $t_m(x)$ would be the positive time when the moving part of the boundary of $\{u^{(m)} \geq 1\}$ crosses K, and $t_\infty(x)$ would be the positive time when the moving part of the boundary of $\{u^{(\infty)} = 1\}$ crosses K.

4 Monotonicity, inclusions, and consequences

We start with a lemma which is a simple consequence of Lemma 4.2 in [K1].

4.1 Lemma (Monotonicity of $u^{(m)}$). *For each $m > 0$ and any $x \in D^c$ we have $u^{(m)}(x,t)$ is an increasing function of time.*

Proof. If $u^{(m)}(x,t) < 1$, then this follows immediately from Lemma 4.2 of [K1]. As a consequence of this fact, we can already say that the $A^{(m)}(t)$ are nested nondecreasing sets. Now we consider the set $S := \{(x,t) \ : \ u^{(m)}(x,t) > 1\}$. By observing that $u^{(m)}$ satisfies the heat equation within S and invoking standard parabolic regularity theory, we can conclude that $u^{(m)}(x,t)$ will assume the boundary values $1 + \frac{p(x)}{m}$ on ∂D continuously, except possibly at the initial corner. By Corollary 1.3 of [AK] the function $(u^{(m)} - 1)_+$ is continuous within D^c, so we can extend $u^{(m)}$ to be 1 on the rest of the parabolic boundary of S in a continuous fashion, except again at $t = 0$ where this set meets ∂D. Now we consider the function

$$U(x,t) := u^{(m)}(x,\ t+\epsilon) - u^{(m)}(x,\ t)\ . \tag{4.1}$$

Within S, this function satisfies the heat equation. On $\partial S \cap \partial D$, we have $U(x,t) \equiv 0$. (If we allow $p(x)$ to also depend on time, but we require it to be nondecreasing in time, then we have $U(x,t) \geq 0$ here which is also fine.

It is not at all clear, however, how to deal with the most general case where $p(x,t)$ is allowed to decrease in time.) On the rest of the parabolic boundary of S we have $U(x,t) = u^{(m)}(x,\ t+\epsilon) - 1 \geq 0$. In the corner where $U(x,t)$ is discontinuous, it stays bounded between zero and $1 + \frac{M}{m}$, and hence by the weak maximum principle, $U(x,t)$ is nonnegative in all of D. Hence

$$\frac{u^{(m)}(x,\ t+\epsilon) - u^{(m)}(x,\ t)}{\epsilon} \tag{4.2}$$

is nonnegative, and the result follows by taking $\epsilon \downarrow 0$.
Q.E.D.

4.2 Lemma (Monotonicity of $u^{(\infty)}$). *$u^{(\infty)}$ is monotone increasing in t for $\mathcal{L}^n$ a.e. x.*

Proof. By the last lemma the $u^{(m)}$ are monotone increasing in t. Now to come to a contradiction, suppose $h > 0$ is a number and $\varphi(x,t)$ is a bounded nonnegative function which is compactly supported in our domain and which satisfies

$$\int\!\!\int \varphi(x,\ t)u^{(\infty)}(x,\ t)\ dx\ dt > \int\!\!\int \varphi(x,\ t-h)u^{(\infty)}(x,\ t)\ dx\ dt\ .$$

Such a construction is possible if the conclusions of our theorem are not satisfied. Then $\psi(x,t) := \varphi(x,\ t) - \varphi(x,\ t-h)$ is an admissible test function which satisfies

$$\int\!\!\int \psi(x,\ t)u^{(\infty)}(x,\ t)\ dx\ dt > 0\ , \quad \text{and} \quad \int\!\!\int \psi(x,\ t)u^{(m)}(x,\ t)\ dx\ dt \leq 0$$

which contradicts the weak-$*$ L^∞ convergence of the $u^{(m)}$ to $u^{(\infty)}$.
Q.E.D.

Now without data on the slot which "competes" with the increasing diffusivity (or if the functions solve the equation on all of $\mathbb{R}^n$), it is immediate by rescaling that the diffusive regions (the $A^{(m)}(t)$) must be nondecreasing with m. In our case, however, we need to produce an appropriate barrier.

4.3 Theorem (Temperature increases with m.). *If $m < k$, then*

$$m[u^{(m)} - 1]_+ \leq k[u^{(k)} - 1]_+\ . \tag{4.3}$$

As a trivial consequence we can say that the $A^{(m)}(t)$ are nested: $m < k$ implies $A^{(m)}(t) \subset A^{(k)}(t)$.

Proof. We make the following definition:

$$v^{(k)}(x,\ t)\ :=\ \begin{cases} u^{(m)}(x,\ t) & \text{if } u^{(m)}(x,\ t) < 1 \\ \\ 1+\frac{m}{k}(u^{(m)}(x,\ t)-1) & \text{if } u^{(m)}(x,\ t) \geq 1 \end{cases} \tag{4.4}$$

and claim that $v^{(k)}(x,\ t)$ is a subsolution of the k-approximating problem. Indeed, this follows very quickly from the following two easily verifiable statements:

1. The set where $v^{(k)} > 1$ is identical to the set where $u^{(m)} > 1$.
2. $k(v^{(k)} - 1)_+ \equiv m(u^{(m)} - 1)_+$.

(Only the definition of $v^{(k)}$ is needed to verify these statements!) By the first observation the free boundaries of $v^{(k)}$ and $u^{(m)}$ are identical in time and space, so the speed of these boundaries at every point is also identical. On the other hand, the second observation quickly leads to the conclusion that the function $v^{(k)}$ satisfies the free boundary condition and the boundary value condition on the slot for the k-approximating problem exactly. Finally, we simply compute:

$$\begin{aligned} v_t^{(k)}(x,\ t) &\leq u_t^{(m)}(x,\ t) \\ &= m\Delta(u^{(m)}(x,\ t)-1)_+ \\ &= k\Delta(v^{(k)}(x,\ t)-1)_+ \end{aligned}$$

which shows that $v^{(k)}$ is a local subsolution to the k-approximating problem, and therefore $v^{(k)}(x,\ t) \leq u^{(k)}(x,\ t)$. (In the first inequality above we have used Lemma (4.1) .) We now have the simple consequence

$$k[u^{(k)}-1]_+ \geq k[v^{(k)}-1]_+ \equiv m[u^{(m)}-1]_+\ , \tag{4.5}$$

which is the monotonicity we require.
Q.E.D.

4.4 Corollary (Pointwise convergence of the temperature). *The sequence of functions $\{m[u^{(m)}-1]_+\}$ converges pointwise almost everywhere to V. In particular, the limiting function V is unique. (No subsequence is ever needed.)*

Proof. By the last theorem combined with the estimates in place already, at each point we have a bounded increasing sequence.
Q.E.D.

4.5 Corollary (Representation of $u^{(\infty)}$). *There is an increasing set-valued function of t which we call $Q(t)$ such that $u^{(\infty)}(x,\ t)$ admits the representation for almost every $(x,\ t)$:*

$$u^{(\infty)}(x,t) = \chi_{Q(t)}(x) + u_I(x)\chi_{Q(t)^c}(x)\ . \tag{4.6}$$

Furthermore, $u^{(\infty)}(x,\ t)$ is the pointwise limit of the functions $u^{(m)}(x,\ t)$ almost everywhere, and $Q(t)$ can be chosen to be equal to the set $\{x \in \mathbb{R}^n\ :\ \tau_\infty(x) < t\}$ ($Q(t)$ is increasing, means in the sense of set inclusion.)

Proof. By Lemma (4.2), we know that $u^{(\infty)}(x,\ t)$ is an increasing function of t and so by using Lemma (4.2) along with Equation (2.3) we can conclude that $u_I(x) \leq u^{(\infty)}(x,\ t) \leq 1$. So, to prove Equation (4.6) it suffices to show that $u^{(\infty)}(x,\ \cdot)$ does not attain values strictly between $u_I(x)$ and 1 for a.e. x. Indeed, by combining the previous theorem with Remark (2.4) we see that the limit

$$\lim_{m\to\infty} u^{(m)}(x_0,\ t_0) \tag{4.7}$$

exists for almost every $(x_0, t_0) \in \mathbb{R}^{n+1}$ and is either equal to 1 or to $u_I(x_0)$. At that point it is a simple exercise to show that this pointwise limit coincides with the weak-$*$ L^∞ limit almost everywhere.

In terms of the "choice" of $Q(t)$, it is clear that the only x where there can be a choice is on the (possibly empty) set $\{x \in \mathbb{R}^n\ :\ u_I(x) = 1\}$. Examination of the definition of $Q(t)$ combined with the monotonicity of $u^{(\infty)}(x,\ t)$ in t makes it clear that $Q(t) := \{x \in \mathbb{R}^n\ :\ \tau_\infty(x) < t\}$ will suffice. (Note that an equally good choice for $Q(t)$ would be to take $Q(t) := \{x \in \mathbb{R}^n\ :\ t_\infty(x) < t\}$.) Q.E.D.

4.6 Remark (Uniqueness of limits of subsequences). Although we had originally needed subsequences to be sure that our $u^{(m)}$ would converge in weak-$*$ L^∞, the last result makes it clear that $u^{(m)}$ will converge both pointwise and weak-$*$ L^∞ to $u^{(\infty)}$ without the need to extract a subsequence.

4.7 Corollary (V is harmonic within the diffusive region). *The spatial Laplacian of V is zero for x in the interior of $A(t)$. As a consequence, if $\partial D \in C^{k,\alpha}$ where $k \geq 2$, then V attaches to the slot D in a $C^{k,\alpha}$ fashion.*

Although the function V is discontinuous in general, we do get regularity in space.

4.8 Theorem (Spatial continuity of V.). *$V(\ \cdot\ ,\ t)$ is continuous for almost every time, t.*

Proof. We will prove that $V(\ \cdot\ ,\ t)$ is continuous by proving that it is both upper and lower semicontinuous. Lower semicontinuity follows immediately from the fact that the functions $m[u^{(m)}(\ \cdot\ ,\ t) - 1]_+$ are continuous and are increasing as functions of m. To get upper semicontinuity we have a little bit more work.

By Lemma (4.2), $d\nu = u_t^{(\infty)}$ is a nonnegative measure. Our Radon measure can be "sliced" into Radon measures of one dimension less for a.e. t. (See [M] p. 139 - 142, and Equation (10.3) on p. 140 in particular.) We will call these slices $d\nu_t$. Since for a.e. t we now have $\Delta_x V(x,t) = d\nu_t$, we know $V(\cdot,t)$ is subharmonic, and therefore upper semicontinuous for those values of t.
Q.E.D.

5 Nontriviality and nondegeneracy

Due to the competition between the decreasing data on ∂D and the increasing diffusivity of our m-approximating problems, we need now to rule out two trivial cases. The possibilities that we need to exclude are:

1. The possibility that the free boundary moves to infinity as soon as $t > 0$.

2. The possibility that the free boundary never moves.

Conceptually, our elimination of these cases is trivial. We simply produce a family of supersolutions (and then subsolutions) to our m-approximating problems (Equation (1.2)) whose free boundaries move in a suitable manner, independent of m, and then we make use of the comparison principles that our equations enjoy (see [K1]).

5.1 Theorem (Boundedness of the free boundary). *If u_I is compactly supported, then so is $u^{(\infty)}(x, t_0)$ for any $t_0 > 0$. In fact, no matter what the initial data, there is always a supersolution whose free boundary has constant speed.*

Note that we need the compact support of the initial data for this result to hold.
Proof. By using the boundedness of D, and by translating and rescaling it will suffice to produce a supersolution to the problem where $D^c = B_1$, $u_I = \chi_{\{B_2 \setminus B_1\}}$, and $p(x) \equiv k$. We claim that the function

$$v^{(m)}(r,t) := \left(\frac{k(2 - r + \ell t)}{m(1 + \ell t)} + 1 \right) \chi_{(1,\ 2+\ell t]}(r) \tag{5.1}$$

will suffice if ℓ is sufficiently large. Note that this function is linear in the radial variable, and that the free boundary moves with speed ℓ for all time.

The following computations are elementary, and they prove our claim that $v^{(m)}$ is a local supersolution:

$$m\Delta(v^{(m)}-1)_+ = \frac{k(1-n)}{r(1+\ell t)} \le 0 \le \frac{k\ell(r-1)}{m(1+\ell t)^2} = v_t^{(m)} \ . \tag{5.2}$$

If we are on the free boundary, so that $r = 2+\ell t$, then

$$\left|\frac{\partial}{\partial r}\left[m(v^{(m)}-1)_+\right]\right| = \frac{k}{(1+\ell t)} \le \ell = \text{speed of the free boundary.} \tag{5.3}$$

Q.E.D.

5.2 Theorem (Nontrivial motion of the free boundary). *The boundary of the diffusive region does not remain stationary, and for any $R > 0$, there exists a time t_0 such that for all $t > t_0$, we have $B_R \subset A(t_0)$.*

5.3 Remark. If we only want to show that the free boundary does not remain stationary, then we can use the same subsolutions as in the proof of Theorem (4.3). On the other hand, the subsolutions we use here are also needed in the proof of the next theorem.

Proof. Using the fact that D contains an open set, and by translating and rescaling it will suffice to produce a subsolution to the problem where (for any $\alpha \ge 0$,) $D^c = B_{1+\alpha}$, $u_I = \chi_{\{B_{2+\alpha}\setminus B_{1+\alpha}\}}$, and $p(x) \equiv k$, and such that the free boundary of this subsolution moves a fixed distance (independent of α) in a finite time $T(\alpha)$.

We will create our subsolution in a couple of steps. First, let $w^{(m)}(r,t) = w_1^{(m)}(r,t) + w_2^{(m)}(r,t)$ where $w_1^{(m)}(r,t)$ solves for each fixed t

$$\begin{aligned} \Delta_x w_1^{(m)}(r,t) &= 0 && \text{in } B_{2+\alpha+\ell t}\setminus B_{1+\alpha} \\ w_1^{(m)}(r,t) &= 1+\frac{k}{m} && \text{on } \partial B_{1+\alpha} \\ w_1^{(m)}(r,t) &= 1 && \text{on } \partial B_{2+\alpha+\ell t} \end{aligned} \tag{5.4}$$

and $w_2^{(m)}(r,t)$ solves (again for each fixed t)

$$\begin{aligned} \Delta_x w_2^{(m)}(r,t) &= \frac{\epsilon}{m} && \text{in } B_{2+\alpha+\ell t}\setminus B_{1+\alpha} \\ w_2^{(m)}(r,t) &= 0 && \text{on } \partial\{B_{2+\alpha+\ell t}\setminus B_{1+\alpha}\} \ . \end{aligned} \tag{5.5}$$

Since the free boundary will be given by $r = 2 + \alpha + \ell t$, and we only need it to move a fixed distance, we will also assume that $\ell t \leq 1$. The functions $w_1^{(m)}$ and $w_2^{(m)}$ can be given explicitly, and their relevant properties are given in the appendix.

By invoking Corollary (8.3) from the appendix, we have

$$-\infty < -\tilde{C}_1(n)k \leq \frac{\partial}{\partial r} m w_1^{(m)}(2 + \alpha + \ell t,\ t) \leq -\tilde{C}_2(n)k < 0 \tag{5.6}$$

where we stress that the constants are independent of α and ℓt. (See the appendix for a few more details, and remember that $0 \leq \ell t \leq 1$.) By the same corollary we can conclude that

$$0 < \tilde{C}_3(n)\epsilon \leq \frac{\partial}{\partial r} m w_2^{(m)}(2 + \alpha + \ell t,\ t) \leq \tilde{C}_4(n)\epsilon < \infty \tag{5.7}$$

again with constants independent of α and ℓt. Now we take $\epsilon > 0$ sufficiently small to ensure that

$$-\infty < -C_1 k \leq \frac{\partial}{\partial r} \left[m w^{(m)}(2 + \alpha + \ell t,\ t) \right] \leq -C_2 k < 0\ . \tag{5.8}$$

Now by taking $\ell \leq C_2 k/2$ we can be sure that our function is a subsolution along the free boundary. We note that $\Delta m (w^{(m)}(r,\ t) - 1)_+ = \epsilon > 0$ in the region where $w^{(m)}(r,\ t) > 1$. Since

$$\lim_{m \to \infty} w_t^{(m)}(r,\ t) = 0\ , \tag{5.9}$$

Once m is sufficiently large, we automatically have

$$\Delta m (w^{(m)}(r,\ t) - 1)_+ \geq w_t^{(m)}(r,\ t)\ . \tag{5.10}$$

Since the free boundary of our subsolution moves with speed $\ell > 0$, we are done.
Q.E.D.

The following theorem shows the instantaneous detachment of the free boundary from the slot, ∂D, even if $u_I(x) \leq \lambda < 1$ in all of D^c.

5.4 Theorem (Instantaneous formation of the diffusive region). *If $t_0 > 0$, The set $A(t_0)$ contains an open neighborhood of ∂D.*

Proof. Because we have assumed that $\partial D \in C^{2,\alpha}$, every point on ∂D can be touched from within with a tangent ball, and then we can use the same subsolutions of the previous theorem to force instantaneous movement of the free boundary.
Q.E.D.

6 Spatial regularity results

In this section we will derive spatial regularity for both the function $V(x,t)$ and for the free boundary. We apply the Baiocchi transformation to $V(x,t)$ and define:

$$W(x,t) := \int_0^t V(x,\ s)\ ds\ . \tag{6.1}$$

Observe that by Lemma (4.2) (which shows that the diffusive regions are increasing in time) and by the positivity of $V(x,t)$ in the diffusive region, it is clear that the set $\{W > 0\}$ is identical to the set $\{V > 0\}$. Now to find the regularity of $\partial\{x \in \mathbb{R}^n\ :\ W(x,T) > 0\}$ we will show that $W(\cdot,t)$ belongs to $H^1_{loc}(D^c)$ for almost every t, and then that $W(x,t)$ is a weak solution of the following obstacle problem in almost every time slice $t = T$

$$0 \leq W(x,T), \qquad \Delta_x W(x,T) = \chi_{\{W(x,T)>0\}}(x)(1 - u_I(x))\ . \tag{6.2}$$

After that we will be able to invoke regularity results for the obstacle problem due to Caffarelli, Kinderlehrer, Nirenberg, and Blank.

For simplicity, we let $\alpha_m(s) := m(s-1)_+$. We start by stating some simple trace results. Basically, we need to adapt Equation (2.1) to some situations with slightly different test functions.

6.1 Lemma (First Trace result). *If $\psi(x) \in C^\infty$ is supported in the interior of D^c, then the following formula holds for a.e. T:*

$$\begin{aligned} &\int_{D^c}\int_0^T [\Delta_x\psi(x)]\ \alpha_m(u^{(m)}(x,t))\ dt\ dx \\ &= \int_{D^c} \psi(x)[u^{(m)}(x,T) - u_I(x)]\ dx\ . \end{aligned} \tag{6.3}$$

Proof. We make the following definition

$$\varphi(x,t) := \begin{cases} \psi(x) & t \leq T \\ \Theta(x,t) & t > T \end{cases} \tag{6.4}$$

where $\Theta(x,t)$ is chosen to ensure that $\varphi(x,t)$ is a permissible test function for our m-approximating problem. In particular, we need $\varphi(x,t) \in C^\infty$ and we need it to converge to zero as $t \to \infty$. Neither requirement poses any difficulty.

By using the trace result of [AK] (see Theorem 1.1 of [AK]) our functions $u^{(m)}(x,t)$ solve our m-approximating problem starting at time T with initial data $u^{(m)}(x,T)$ for almost every T, and so (for those T) we can use $\Theta(x,t)$ as

the test function in Equation (2.1) to obtain:

$$\begin{aligned}
& \int_{D^c}\int_T^\infty \Theta_t(x,t)u^{(m)}(x,t)\ dt\ dx \\
+ & \int_{D^c}\int_T^\infty \Delta_x\Theta(x,t)\alpha_m(u^{(m)}(x,t))\ dt\ dx \qquad (6.5)\\
= - & \int_{D^c} \psi(x)u^{(m)}(x,T)\ dx\ .
\end{aligned}$$

(Note that $\Theta(x,T) = \psi(x)$ since we have required that $\varphi(x,t)$ be smooth.) Now by subtracting this equation from what we have when we plug in the function $\varphi(x,t)$ defined in Equation (6.4) into Equation (2.1) we get Equation (6.3) immediately.
Q.E.D.

The proof of the following result is almost identical to the proof of the trace result above, so we omit it.

6.2 Lemma (Second Trace result). *For a.e.* t_0, t_1 *such that* $0 \leq t_0 < t_1 < \infty$, *and for* $\varphi \in C^\infty(\mathbb{R}^n \times [t_0,t_1])$, *which satisfies* $\varphi \equiv 0$ *on an open set containing* $D \times [t_0,t_1]$, *we have*

$$\begin{aligned}
& \int_{D^c}\int_{t_0}^{t_1} \varphi_t(x,t)u^{(m)}(x,t)\ dt\ dx + \int_{D^c}\int_{t_0}^{t_1} \Delta_x\varphi(x,t)\alpha_m(u^{(m)}(x,t))\ dt\ dx \\
& = \int_{D^c}\left(\left[\varphi(x,s)u^{(m)}(x,s)\right]\Big|_{s=t_0}^{s=t_1}\right)\ dx\ .
\end{aligned} \tag{6.6}$$

Now we state the standard energy estimate for our situation.

6.3 Lemma (Energy estimates for the $u^{(m)}$). *Let* $0 < r < R$ *and* $0 \leq t_0 < t_1$. *Then there is a constant of the form*

$$C = \frac{C(n)}{(R-r)^2} \tag{6.7}$$

such that the following energy estimate holds:

$$\int_{t_0}^{t_1}\int_{B_r(x_0)} |\nabla\alpha_m(u^{(m)})|^2\ dx\ dt \leq C\int_{t_0}^{t_1}\int_{B_R(x_0)} \alpha_m(u^{(m)})^2\ dx\ dt\ . \tag{6.8}$$

6.4 Remark (Independence of Time). Notice that the constant C is independent of time and notice that the time intervals in the integrals in each side of the inequality are identical.

Proof. We choose $\eta(x) \in C_0^\infty(B_R)$ such that

1. $\eta \equiv 1$ on $\overline{B}_r$,
2. $0 \le \eta \le 1$, and
3. $|\nabla\eta| \le 4(R-r)^{-1}$.

Now let $\varphi(x,t) := \alpha_m(u^{(m)})\eta(x)^2$, and apply the last lemma and Green's identity to obtain:

$$\begin{aligned}&\int_{B_R}\int_{t_0}^{t_1} \varphi_t(x,t)u^{(m)}(x,t)\;dt\;dx - \int_{B_R}\left(\left[\varphi(x,s)u^{(m)}(x,s)\right]\Big|_{s=t_0}^{s=t_1}\right)\;dx \\ &= \int_{B_R}\int_{t_0}^{t_1} \nabla_x\varphi(x,t)\nabla_x\alpha_m(u^{(m)})\;dt\;dx\;.\end{aligned} \tag{6.9}$$

The fact that we can apply Green's identity above can be justified using Lemma 1.2 of [AK]. By Lemma (4.1) we know that for a.e. $x \in D^c$, $\mu_x(t) := u_t(x,t)$ is a non-negative Radon measure, whence it suffices to have $\phi(t) \in C_0(0,+\infty)$ for the distributional pairing $(\mu_x(t), \phi(t))$ to be defined, and non-negative if $\phi \ge 0$. Using Equation (2.1) it is easy to see that the function $g(x) := (\mu_x(t), \chi_{[t_0,t_1]}(t))$ is locally integrable in x. In other words, we can integrate the left hand side of Equation (6.9) by parts in time to give us the following inequality:

$$\begin{aligned}&\int_{B_R}\int_{t_0}^{t_1} \nabla_x\left(\alpha_m(u^{(m)})\eta^2(x)\right)\nabla_x\alpha_m(u^{(m)})\;dt\;dx \\ &= -\int_{B_R}\left[\int_{t_0}^{t_1}\left(\alpha_m(u^{(m)})\eta^2(x)\right)d\mu_x(t)\right]\;dx \\ &\le 0.\end{aligned}$$

This inequality implies

$$\int_{B_R}\int_{t_0}^{t_1} \eta^2\left|\nabla_x\alpha_m(u^{(m)})\right|^2\;dt\;dx \le$$

$$2\int_{B_R}\int_{t_0}^{t_1}\left(\eta\left|\nabla_x\alpha_m(u^{(m)})\right|\right)\left(\alpha_m(u^{(m)})\left|\nabla_x\eta\right|\right)\;dt\;dx$$

and so Cauchy-Schwarz gives

$$\begin{aligned}&\int_{t_0}^{t_1}\left[\int_{B_R}\eta(x)^2\left|\nabla_x\alpha_m(u^{(m)})\right|^2dx\;\right]dt \\ &\le 4\int_{t_0}^{t_1}\left[\int_{B_R}\left|\nabla_x\eta(x)\right|^2\alpha_m(u^{(m)})^2dx\;\right]dt\;.\end{aligned} \tag{6.10}$$

Q.E.D.

The aforementioned independence of time of the constant C leads immediately to the following theorem:

6.5 Theorem (Time-slice energy estimates). *For almost every time t we have*

$$\int_{B_r(x_0)} |\nabla \alpha_m(u^{(m)}(x,t))|^2 \, dx \leq C \int_{B_R(x_0)} \alpha_m(u^{(m)}(x,t))^2 \, dx \, , \tag{6.11}$$

and also

$$\int_{B_r(x_0)} |\nabla V(x,t)|^2 \, dx \leq C \int_{B_R(x_0)} V(x,t)^2 \, dx \, , \tag{6.12}$$

where once again, the constant C has the form given in Equation (6.7).

Proof. Equation (6.11) is already known from the previous lemma, and Equation (6.12) follows from Equation (6.11) by using the L^2 convergence of $\alpha_m(u^{(m)}(x,t))$ to $V(x,t)$ and by using the lower semicontinuity of the Dirichlet integral.
Q.E.D.

6.6 Corollary (Energy estimates for $V(\cdot,t)$). *For almost every t, $V(\cdot,t) \in H^1_{loc}(D^c)$.*

6.7 Theorem (Energy estimates for $W(\cdot,t)$). *For all $t > 0$, $W(\cdot,t) \in H^1_{loc}(D^c)$.*

Proof. Let K be a compact subset of D^c. Since $W(\cdot,t) \leq Mt$, we have $W(\cdot,t) \in L^\infty(K) \subset L^2(K)$. It remains to show that $\frac{\partial W}{\partial x_i}(\cdot,t) \in L^2(K)$.

Let $\varphi \in C_0^1(K)$ and estimate.

$$
\begin{aligned}
\left(\frac{\partial W}{\partial x_i},\ \varphi\right) &= -\int_K W(x,t)\frac{\partial \varphi}{\partial x_i}(x)\,dx \\
&= -\int_K\int_0^t V(x,\ s)\frac{\partial \varphi}{\partial x_i}(x)\,ds\,dx \\
&= -\int_K\int_0^t \lim_{m\to+\infty}\alpha_m(u^{(m)}(x,\ s))\,\frac{\partial \varphi}{\partial x_i}(x)\,ds\,dx \\
&= -\lim_{m\to+\infty}\int_K\int_0^t \alpha_m(u^{(m)}(x,\ s))\,\frac{\partial \varphi}{\partial x_i}(x)\,ds\,dx \\
&= \lim_{m\to+\infty}\int_K\int_0^t \frac{\partial}{\partial x_i}\left(\alpha_m(u^{(m)}(x,\ s))\right)\varphi(x)\,ds\,dx \\
&= \lim_{m\to+\infty}\int_K \varphi(x)\left[\int_0^t \frac{\partial}{\partial x_i}\left(\alpha_m(u^{(m)}(x,\ s))\right)\,ds\right]\,dx
\end{aligned}
$$

where we have used Lebesgue's Dominated Convergence Theorem repeatedly in the computation above. Now, by using Minkowski's integral inequality and the Cauchy-Schwarz Inequality we have

$$
\begin{aligned}
&\left|\int_K \varphi(x)\left[\int_0^t \frac{\partial}{\partial x_i}\left(\alpha_m(u^{(m)}(x,\ s))\right)\,ds\right]\,dx\right| \\
&\le \left|\int_K \varphi(x)^2\,dx\right|^{1/2}\left|\int_K\left(\int_0^t \frac{\partial}{\partial x_i}\left(\alpha_m(u^{(m)}(x,\ s))\right)\,ds\right)^2 dx\right|^{1/2} \\
&\le ||\varphi||_{L^2(K)}\int_0^t\left|\int_K\left(\frac{\partial}{\partial x_i}\left(\alpha_m(u^{(m)}(x,\ s))\right)\right)^2 dx\right|^{1/2} ds \\
&\le ||\varphi||_{L^2(K)}\left|\int_0^t\int_K\left(\frac{\partial}{\partial x_i}\left(\alpha_m(u^{(m)}(x,\ s))\right)\right)^2 dx\,ds\right|^{1/2}\cdot\left[\int_0^t ds\right]^{1/2} \\
&\le t^{1/2}||\varphi||_{L^2(K)}||\nabla_x\alpha_m(u^{(m)}(x,\ s))||_{L^2(K\times(0,t))} \\
&\le Ct^{1/2}||\varphi||_{L^2(K)}
\end{aligned}
$$

where the last constant is independent of m by Lemma (6.3).
Q.E.D.

6.8 Remark. In fact, once we show that $W(\cdot,t)$ is a solution of the obstacle problem, we will be able to infer from elliptic regularity theory, that $W(\cdot,t)$ is $C^{1,\alpha}$ in space for all $\alpha<1$.

In order to derive Equation (6.2) we will need to commute the Laplacian with the integral in time. To accomplish this commutation we turn back to the approximating problem where this switch is simpler.

By using weak-$*$ L^∞ convergence of the temperature functions we have

$$\begin{aligned}
&\lim_{m\to\infty}\int_{D^c}\int_0^T [\Delta_x\psi(x)]\ m\left[u^{(m)}(x,t)-1\right]_+\ dt\ dx\\
&=\int_{D^c}\int_0^T [\Delta_x\psi(x)]\ V(x,t)\ dt\ dx\\
&=\int_{D^c}[\Delta_x\psi(x)]W(x,T)\ dx\ .
\end{aligned}$$

On the other hand, by using the dominated convergence theorem we conclude that for almost every T

$$\begin{aligned}
&\lim_{m\to\infty}\int_{D^c}\psi(x)[u^{(m)}(x,T)-u_I(x)]\ dx\\
&=\int_{D^c}\psi(x)[u^{(\infty)}(x,T)-u_I(x)]\ dx\\
&=\int_{D^c}\psi(x)\chi_{\{W(x,T)>0\}}(x)(1-u_I(x))
\end{aligned}$$

Since $W\geq 0$, we can combine the last two computations with the previous lemma to conclude that $W(\cdot,\ t)$ solves the obstacle problem for a.e. $t>0$. Equation (6.2) and this fact allows us to use the technology of [Bl] and [C] to infer regularity of the free boundary here as long as we satisfy the nondegeneracy condition:

$$u_I(x)\leq\lambda<1\ . \tag{6.13}$$

(u_I satisfying Equation (6.13) will be referred to as *nondegenerate initial data.*) In this case we will have the next theorem which we state after one simple definition.

6.9 Definition (Minimum diameter). The minimum diameter of a set $S\subset\mathbb{R}^n$ (denoted "$m.d.(S)$") is the infimum of the distances between parallel hyperplanes enclosing S.

6.10 Theorem (Regularity of the free boundary in space). *Assume u_I is continuous, nondegenerate initial data. Then there is a modulus of continuity σ which depends on λ, n, and the modulus of continuity of u_I such that at almost any time t, (indeed every time t_0 where Equation (6.2) is valid) and for any free boundary point (x_0,t_0) contained in the interior of D^c we have either (with $B_r(x)$ denoting the (spatial) ball centered at x with radius r)*

$$m.d.(B_r(x_0)\cap A(t)^c)\leq r\sigma(r)\quad \textit{for all } r\leq 1\ , \tag{6.14}$$

(so that the nondiffusive region is "cusp-like") or

$$\lim_{r \to 0} \frac{|B_r(x_0) \cap A(t)|}{|B_r|} = \frac{1}{2} \tag{6.15}$$

in which case we will say that (x_0, t_0) is a regular point of the free boundary.

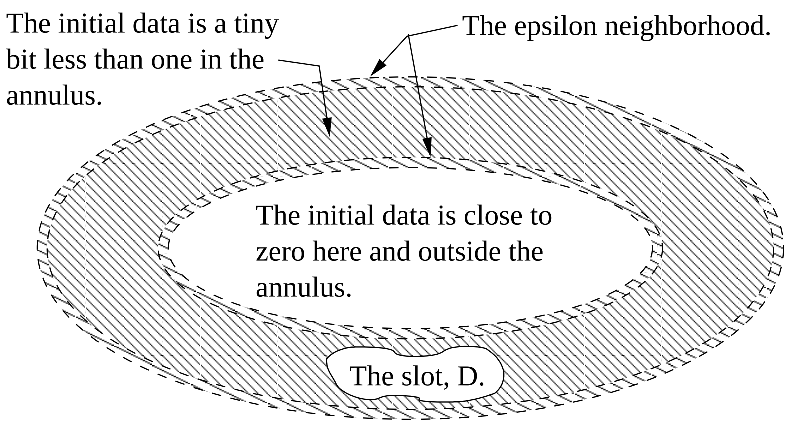
Proof. Simply use Equation (6.2) together with the results from [Bl].
Q.E.D.

6.11 Theorem (Better regularity). *Assume the hypotheses of the previous theorem and assume that (x_0, t_0) is a regular point of the free boundary. Then the free boundary intersected with $\{t = t_0\}$ will be Reifenberg vanishing near x_0. (For a definition of Reifenberg vanishing, see [Bl].) If u_I is Dini continuous, then the free boundary will be C^1 (in space) near x_0, and if $u_I \in C^{k,\alpha}$ then the free boundary will be $C^{k+1,\alpha}$ (in space) near x_0.*

Proof. Again, just combine Equation (6.2) with the results from [Bl].
Q.E.D.

6.12 Remark (Clarification). When we say "near" x_0 in the theorem above we mean near in space only. In other words, we are specifically talking about the free boundary restricted to the time slice $t = t_0$.

At this point we need to call attention to our assumptions that u_I is continuous and nondegenerate. Indeed it is well known that there are examples of "persistent corners" in Hele-Shaw problems. (See [KLV] for example.) Our theorem does not contradict this fact. Our spatial regularity theorem says nothing if u_I is discontinuous, or if the set $\{u_I = 1\} \cap D^c$ is nonempty. On the other hand, our assumptions do not rule out "focusing" or changes in topology of the diffusive region, so it is certainly nontrivial that corners do not arise in this setting. Consider a case where u_I is very close to zero except for an ϵ neighborhood of an annulus which contains the slot. In this annulus assume that u_I is extremely close to one. (The epsilon neighborhood is needed to make u_I continuous.) In this case, the diffusive region will make its way around the annulus in each direction very quickly, but expand slowly into the region inside and outside of the annulus. Thus, it will meet itself on the other side of the annulus long before it fills in the interior.



7 Continuity and dealing with all times

We wish now to improve the results from the previous section by extending them from almost every time to every time. Indeed, this prevents corners from arising as "transitions" from the situations which are permissible for sets of time with positive measure. (Consider for example the sets $S(t) := \{(x,y) : xy < t-1$ for the interval $t \in [0,2]$. By the results of the last section, it would still be possible for $S(t)$ to be the diffusive region for a Hele-Shaw flow, since the corner only occurs when t belongs to the zero measure set $\{1\}$.) Of course in this entire section we make the standing assumption that u_I is nondegenerate initial data (see Equation (6.13)). We start with a simple lemma summarizing some of the regularity we have for W.

7.1 Lemma (Regularity for W). *$W(x,t)$ is continuous in space and continuous and convex and nondecreasing in time. For almost every time, T, $W(x,T)$ satisfies Equation (6.2) with boundary data*

$$W(x,T) = p(x)T \qquad x \in \partial D \; . \tag{7.1}$$

Proof. By using the maximum principle together with the fact that the diffusive region increases with time, we see that $V(x,t)$ must be an increasing function of time. Now convexity of W follows from this fact and from the definition of W. The rest of the lemma follows immediately from the definition of W and the spatial continuity of $V(x,t)$ along with its boundary data on ∂D.
Q.E.D.

7.2 Theorem (Measure of the diffusive region). *There exists a constant $C = C(n, \alpha, \partial D)$ such that if $0 \le t - s \le 1$, then*

$$|A(t) \setminus A(s)| \le C||p||_{C^{2,\alpha}(\partial D)} \left(\frac{t-s}{1-\lambda} \right) . \tag{7.2}$$

(Recall that $A(t)$ is the diffusive region at time t, and for $S \subset \mathbb{R}^n$, we let $|S|$ denote the Lebesgue n-dimensional measure of S.)

Proof. We adapt the proof of Theorem 4.1 of [Bl] to the current setting. Fix t and s. Because the diffusive regions are nested, and because Equation (6.2) holds for almost every time, without loss of generality we can assume that it holds for both t and s. Let $L := A(t) \setminus A(s)$, let $\Psi(x) := W(x,t) - W(x,s)$ and observe that $\Delta\Psi(x) = \chi_{_L}(1 - u_I) \ge 0$, and

$$\begin{array}{ll} \Psi(x) = 0 & x \in FB(t), \\ \Psi(x) = (t-s)p(x) & x \in \partial D \, . \end{array} \tag{7.3}$$

From this fact and by the weak maximum principle, it follows that

$$0 \le W(x,t) - W(x,s) \le (t-s)||p||_{L^\infty(\partial D)} \tag{7.4}$$

for all $x \in A(t)$. (Nonnegativity is actually a consequence of the previous lemma.)

Observe that $W(x,t) - W(x,s)$ is harmonic within $A(s)$ so that

$$\begin{aligned} 0 &= \int_{A(s)} \Delta(W(x,t) - W(x,s))dx \\ &= \int_{\partial D} \frac{\partial}{\partial \nu}(W(x,t) - W(x,s))d\mathcal{H}^{n-1} - \int_{FB(s)} \frac{\partial}{\partial \nu}(W(x,t) - W(x,s))d\mathcal{H}^{n-1} \\ &= \int_{\partial D} \frac{\partial}{\partial \nu}(W(x,t) - W(x,s))d\mathcal{H}^{n-1} - \int_{FB(s)} \frac{\partial}{\partial \nu}W(x,t)d\mathcal{H}^{n-1} \end{aligned}$$

Now by using boundary regularity for harmonic functions combined with Equation (7.4) we can conclude

$$\left| \int_{\partial D} \frac{\partial}{\partial \nu}(W(x,t) - W(x,s))d\mathcal{H}^{n-1} \right| \le C(n, \alpha, \partial D)(t-s)||p||_{C^{2,\alpha}(\partial D)} \tag{7.5}$$

By combining this fact with the last computation, we conclude that

$$\left| \int_{FB(s)} \frac{\partial}{\partial \nu}W(x,t)d\mathcal{H}^{n-1} \right| \le C(n, \alpha, \partial D)(t-s)||p||_{C^{2,\alpha}(\partial D)} \, . \tag{7.6}$$

On the other hand we have

$$\begin{aligned}(1-\lambda)|L| &\leq \int_L (1-u_I)\,dx \\ &= \int_L \Delta W(x,t)\,dx \\ &= \int_{\partial L} \frac{\partial}{\partial \nu} W(x,t) d\mathcal{H}^{n-1} \\ &= \int_{FB(s)} \frac{\partial}{\partial \nu} W(x,t) d\mathcal{H}^{n-1}\end{aligned}$$

which we can combine with Equation (7.6) to give us what we need.
Q.E.D.

7.3 Corollary (Continuity in L^p). *Under the assumptions made at the beginning of this section, the map from t to the function*

$$\chi_{\{W(x,t)>0\}}(1-u_I)$$

is a continuous function from $\mathbb{R}$ into $L^p(D^c)$ for $1 \leq p < \infty$.

7.4 Corollary (Spatial regularity for every time). *All of the results of Theorems (6.10) and (6.11) hold for every time.*

Proof. It suffices to show that Equation (6.2) holds for all time. Fix $\tilde{t} > 0$ and let $t_n \to \tilde{t}$ with t_n chosen so that Equation (6.2) holds at each t_n. Now take a ball, B_R which is large enough to contain $A(\tilde{t})$ in its interior, and let $\Omega := B_R \setminus D$. For each n we let $w_n(x)$ solve the boundary value problem

$$\begin{aligned} &\Delta w_n(x) = \chi_{A(t_n)}(1-u_I) && \text{in } \Omega \\ &w_n(x) = t_n p(x) && \text{on } \partial D \\ &w_n(x) = 0 && \text{on } \partial B_R\,. \end{aligned} \tag{7.7}$$

By standard uniqueness results, $w_n(x) \equiv W(x,t_n)$, as they satisfy the same boundary value problem. By standard elliptic regularity theory, since the boundary data on ∂D will converge to $\tilde{t}p(x)$ and by the last corollary the right hand side of the equation will converge in $L^p(\Omega)$ to $\chi_{A(\tilde{t})}(1-u_I)$, we can conclude that w_n will converge to a function $\tilde{w}$ which satisfies

$$\begin{aligned} &\Delta \tilde{w}(x) = \chi_{A(\tilde{t})}(1-u_I) && \text{in } \Omega \\ &\tilde{w}(x) = \tilde{t}p(x) && \text{on } \partial D \\ &\tilde{w}(x) = 0 && \text{on } \partial B_R\,. \end{aligned} \tag{7.8}$$

On the other hand, $w_n(x) = W(x,t_n)$ converges to $W(x,\tilde{t})$ by the continuity of W in time.
Q.E.D.

8 Appendix

Here we collect some facts we need to construct our subsolutions in the proof of Theorem (5.2). We let $u(r;\alpha,\beta)$ denote the solution to:

$$\begin{aligned} \Delta_x u(r;\alpha,\beta) &= 0 \qquad \text{in } B_{1+\alpha+\beta}\setminus B_\alpha \\ u(r;\alpha,\beta) &= 1 \qquad \text{on } \partial B_\alpha \\ u(r;\alpha,\beta) &= 0 \qquad \text{on } \partial B_{1+\alpha+\beta} \end{aligned} \tag{8.1}$$

and we let $v(r;\alpha,\beta)$ denote the solution to:

$$\begin{aligned} \Delta v(r;\alpha,\beta) &= 2n \qquad \text{in } B_{1+\alpha+\beta}\setminus B_\alpha \\ v(r;\alpha,\beta) &= 0 \qquad \text{on } \partial\{B_{1+\alpha+\beta}\setminus B_\alpha\}\ . \end{aligned} \tag{8.2}$$

We will always assume that $\alpha \geq 1$ and $0 \leq \beta \leq 1$.

8.1 Lemma (Explicit Forms of Our Comparison Functions). *If $n > 2$, then u has the explicit form*

$$u(r;\alpha,\beta) = \frac{r^{2-n} - (1+\alpha+\beta)^{2-n}}{\alpha^{2-n} - (1+\alpha+\beta)^{2-n}}\ , \tag{8.3}$$

and v has the explicit form

$$v(r;\alpha,\beta) = (r^2-\alpha^2) + \frac{(\alpha^2 - (1+\alpha+\beta)^2)(r^{2-n} - \alpha^{2-n})}{(1+\alpha+\beta)^{2-n} - \alpha^{2-n}}\ . \tag{8.4}$$

If $n = 2$, then u has the explicit form

$$u(r;\alpha,\beta) = \left[\log\left(\frac{r}{1+\alpha+\beta}\right)\right] \Big/ \left[\log\left(\frac{\alpha}{1+\alpha+\beta}\right)\right]\ , \tag{8.5}$$

and v has the explicit form

$$v(r;\alpha,\beta) = (r^2-\alpha^2) + \frac{\log\left(\dfrac{r}{\alpha}\right)(\alpha^2 - (1+\alpha+\beta)^2)}{\log\left(\dfrac{1+\alpha+\beta}{\alpha}\right)}\ . \tag{8.6}$$

8.2 Lemma (Derivatives on the Outer Boundaries). *If $n > 2$, then*

$$u_r(1+\alpha+\beta;\alpha,\beta) = \frac{(2-n)(1+\alpha+\beta)^{1-n}}{\alpha^{2-n} - (1+\alpha+\beta)^{2-n}} < 0\ , \tag{8.7}$$

and

$$v_r(1+\alpha+\beta;\alpha,\beta) = 2(1+\alpha+\beta)+(n-2)(1+\alpha+\beta)\frac{(1+\alpha+\beta)^2-\alpha^2}{(1+\alpha+\beta)^{2-n}-\alpha^{2-n}} > 0 \,. \tag{8.8}$$

If $n = 2$, *then*

$$u_r(1+\alpha+\beta;\alpha,\beta) = \frac{1}{(1+\alpha+\beta)\log\left(\dfrac{\alpha}{1+\alpha+\beta}\right)} < 0 \,, \tag{8.9}$$

and

$$v_r(1+\alpha+\beta;\alpha,\beta) = 2(1+\alpha+\beta)+\frac{\alpha^2-(1+\alpha+\beta)^2}{(1+\alpha+\beta)\log\left(\dfrac{1+\alpha+\beta}{\alpha}\right)} > 0 \,. \tag{8.10}$$

Also, for any $n \geq 2$, *we have*

$$\lim_{\alpha\to\infty} u_r(1+\alpha+\beta;\alpha,\beta) = \frac{-1}{1+\beta} < 0 \,, \tag{8.11}$$

and

$$\lim_{\alpha\to\infty} v_r(1+\alpha+\beta;\alpha,\beta) = n(1+\beta) > 0 \,. \tag{8.12}$$

8.3 Corollary (Bounds on the Outer Boundaries). *For* $0 \leq \beta \leq 1$ *and* $\alpha \geq 1$, *there exist constants* γ_i *which are all independent of* α *and* β *such that*

$$-\gamma_1 \leq u_r(1+\alpha+\beta;\alpha,\beta) \leq -\gamma_2 < 0 \,, \tag{8.13}$$

and

$$0 < \gamma_3 \leq v_r(1+\alpha+\beta;\alpha,\beta) \leq \gamma_4 < \infty \,. \tag{8.14}$$

Proof. The proof for u_r is essentially the same as the proof for v_r, so we will only deal with u_r. Because the limit as $\alpha \to \infty$ is strictly negative by the previous lemma there is a large α_0 such that for $\alpha \geq \alpha_0$ we have the desired lower and upper bounds. Next we simply use compactness for the rest of the strip.
Q.E.D.

References

[AK] D. Andreucci and M. Korten, Initial traces of solutions to a one-phase Stefan problem in an infinite strip, *Rev. Mat. Iberoamericana,* 9(1993), no. 2, 315–332.

[BF] B. Bazaliy and A. Friedman, Global existence and asymptotic stability for an elliptic-parabolic free boundary problem: An application to a model of tumor growth, *Indiana Univ. Math. J.,,* 52(2003), no. 5, 1265–1304.

[BBH] Ph. Bénilan, L. Boccardo, and M.A. Herrero, On the limit of solutions of $u_t = \Delta u^m$ as $m \to \infty$, *Rend. Mat. Univ. Pol. Torino,* Fascicolo Speciale (1989), Nonl. PDE's, 1–13.

[BEG] Ph. Bénilan, L.C. Evans, R.F. Gariepy, On some singular limits of homogeneous semigroups, *J. Evol. Eq.,* 3(2003), no. 2, 203–214.

[Bl] I. Blank, Sharp results for the regularity and stability of the free boundary in the obstacle problem, *Indiana Univ. Math. J.* 50(2001), no. 3, 1077–1112.

[Bo] J.E. Bouillet, Nonuniqueness in L^∞: An example, *Lecture notes in pure and applied mathematics,* Vol. 148, Marcel Dekker, Inc., (1993).

[BKM] J.E. Bouillet, M.K. Korten, and V. Márquez, Singular limits and the "Mesa" problem, *Rev. Un. Mat. Argentina,* 41(1998), no. 1, 27–40.

[C] L.A. Caffarelli, The obstacle problem revisited, *J. Fourier Anal. Appl.,* 4(1998), no. 4-5, 383–402.

[CF] L.A. Caffarelli and A. Friedman, Asymptotic behavior of solutions of $u_t = \Delta u^m$ as $m \to \infty$, *Indiana Univ. Math. J.,* 36(1987), no. 4, 203–224.

[DK] B.E.J. Dahlberg and C.E. Kenig, Weak solutions of the porous medium equation, *Trans. Am. Math. Soc.,* 336(1993), no. 2, 711–725.

[DL] P. Daskalopoulos and K. Lee, All time smooth solutions of the one-phase Stefan problem and the Hele-Shaw flow, *Comm. PDE,* 29(2004), no. 1-2, 71–88.

[DB] E. DiBenedetto, Continuity of weak solutions to certain singular parabolic equations, *Ann. Mat. Pura Appl.,* (4), CXXX(1982), 131–176.

[DF] E. DiBenedetto and A. Friedman, The ill-posed Hele-Shaw model and the Stefan problem for supercooled water, *Trans. Amer. Math. Soc.,* 282(1984), no. 1, 183–204.

[EHKO] C.M. Elliott, M.A. Herrero, J.R. King, and J.R. Ockendon, The mesa problem: Diffusion patterns for $u_t = \nabla \cdot (u^m \nabla u)$ as $m \to +\infty$, *IMA J. Appl. Math.,* 37(1986), no. 2, 147–154.

[FH] A. Friedman and S. Huang, Asymptotic behavior of solutions of $u_t = \Delta \phi_m(u)$ as $m \to \infty$ with inconsistent initial values, *Analyse mathématique et applications,* Gauthier-Villars, Paris, (1988), 165–180.

[GQ] O. Gil and F. Quirós, Convergence of the porous medium equation to Hele-Shaw, *Nonlin. Anal.,* 44(2001), no. 8, Ser. A: Theory Methods, 1111–1131.

[I] N. Igbida, The mesa-limit of the porous-medium equation and the Hele-Shaw problem, *Diff. Int. Eqns.,* 15(2002), no. 2, 129–146.

[KT] C.E. Kenig and T. Toro, Free boundary regularity for harmonic measures and Poisson Kernels, *Ann. of Math.,* (2)150(1999), no. 2, 369–454.

[KN] D. Kinderlehrer and L. Nirenberg, Regularity in free boundary problems, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.,* (4)4(1977), no. 2, 373–391.

[KLV] J.R. King, A.A. Lacey, and J.L. Vazquez, Persistence of corners in Hele-Shaw flow, *Euro. J. Appl. Math.,* 6(1995), no. 5, 455–490.

[K1] M.K. Korten, Non-negative solutions of $u_t = \Delta(u-1)_+$: Regularity and uniqueness for the Cauchy problem, *Nonl. Anal., Th., Meth., and Appl.,* 27(1996), no. 5, 589–603.

[K2] M.K. Korten, A Fatou theorem for the equation $u_t = \Delta(u-1)_+$, *Proc. Am. Math. Soc.,* 128(1999), no. 2, 439–444.

[K3] M.K. Korten, On the rectifiability of the free boundary of the one phase Stefan problem, to appear in *Comm. Anal. and Geom.*

[KM] M.K. Korten and C.N. Moore, The two phase Stefan problem: Regularity of solutions to the Cauchy problem, submitted.

[M] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces, *Cambridges studies in advanced mathematics,* Vol. 44, Cambridge University Press, (1995).

[MR] J.A. McGeough and H. Rasmussen, On the derivation of the quasi-steady model in electrochemical machining, *J. Inst. Math. Appl.,* 13(1974), 13–21.

Department of Mathematics
Worcester Polytechnic Institute
Worcester, MA 01609
E-mail address: blanki@wpi.edu

Department of Mathematics
Kansas State University
Manhattan, KS 66506
E-mail address: marianne@math.ksu.edu, cnmoore@math.ksu.edu